Design and Optimization of Degree-Constrained Social Networks

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Land Acknowledgement

- The land on which we gather is the traditional unceded territory of the Wəlastəkwiyik (Maliseet), covered by the "Treaties of Peace and Friendship" which Wəlastəkwiyik (Maliseet), Mi'kmaq and Passamaquoddy Peoples first signed with the British Crown in 1726

- The treaties did not deal with surrender of lands and resources but in fact recognized Mi'kmaq and Wəlastəkwiyik (Maliseet) title and established the rules for what was to be an ongoing relationship between nations

- Furthermore, much of this research was conducted in Sackville, NB (the unceded ancestral lands of the Mi'kmaw people) and Montreal, QC (the traditional territory of the Kanien'kehà:ka)

What Are Socioeconomic Networks?

- **Socioeconomic networks** are **graphs** in which nodes represent agents/infrastructure/etc. and edges represent ties/links
- Networks are ubiquitous in society and the economy: social media, transportation systems, electrical grids, supply chains
- Designing and optimizing networks for **efficiency** and **connectivity** is a focus in computational and algorithmic economics

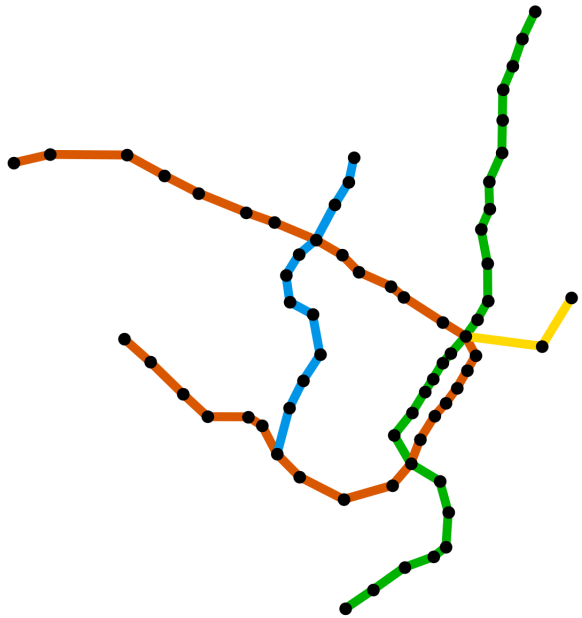


Figure: The **Montreal Metro** represented as a network/graph, drawn to scale.

What Are Communication Networks?

- Consider a specific type of network. Fix $n \geq 3$ and $X \in (0,1)$, then:
  - A (max-strength) **communication network** is a graph G with n agents in which each agent forms one weight-X edge to another
  - Although edge formation is directed, "value" flows both ways between agents and **decays exponentially** with distance
  - Assume no co-sponsorship of edges, so each connected component is **unicyclic**: a cycle with pendant trees attached to it
- **Payoff** of agent u ("how close" it is to all other agents):

$$\varphi_G(u) := \sum_{v \in V(G) \setminus \{u\}} X^{d_G(u,v)}$$

- **Value** of network G (for chemists, twice its Hosoya polynomial at X):

$$\mathcal{V}(G) := \sum_{u \in V(G)} \varphi_G(u) = 2 \sum_{\{u,v\} \subseteq V(G),\, u \neq v} X^{d_G(u,v)} = 2H(G,X)$$

Degree-Unconstrained Networks

- **Efficient:** Network maximizes $\mathcal{V}(G)$ given fixed **constraints** n and X
- **Nash:** Every agent maximizes payoff given all other agents' strategies
- Traditionally, node degree (both incoming and outgoing) is unconstrained; efficient networks thus have very **low diameter**

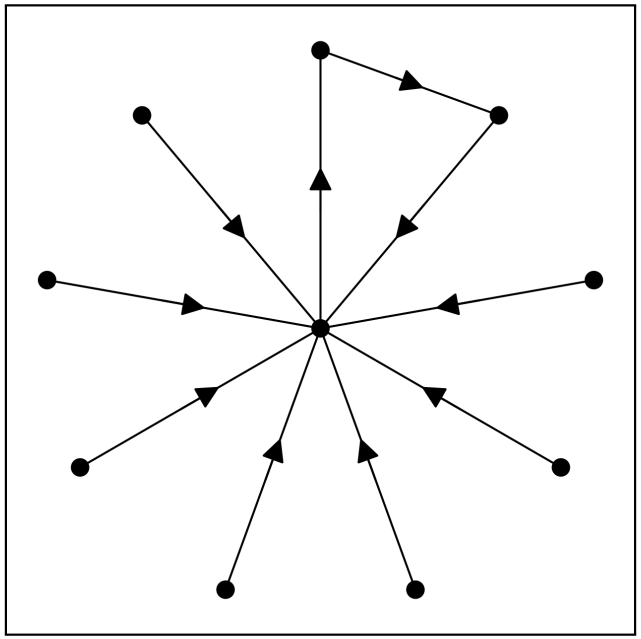


Figure: For all $n \geq 6$ and $X \in (0, 1)$: efficient & connected $\iff$ **star plus edge** [Kak25]. (We show via partitions that this beats disconnected networks too.)

Imposing Degree Constraints

- **Real world:** What if a "central" agent malfunctions? What if a person/firm/station is not capable of relaying so much data?
- Impose a **degree constraint**: a cap $D \in \{2, 3, \ldots, n-2\}$
- Now, larger cycles with many small pendant trees are efficient (cf. a star plus an edge: a small cycle with a large height-1 tree)

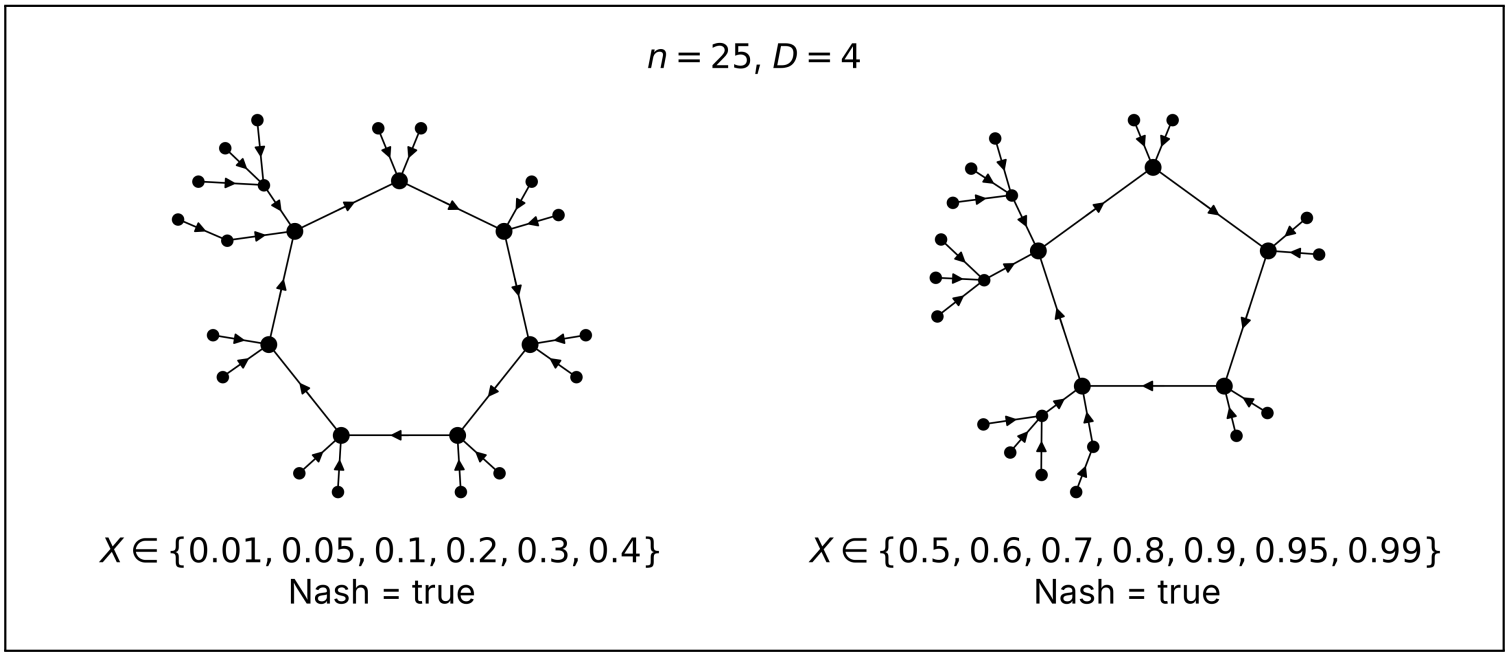


Figure: The two efficient networks for $n = 25$ and $D = 4$.

Computational Challenges

- **Many Nash** networks per fixed (n, D, X), so **not** interested in finding all (but can quickly check individual networks in $\Theta(n^3)$ time)
- Only a **few efficient** ones, which we **do** want, but difficult to find:
  - Infinitely many $X \in (0, 1)$ per fixed (n, D) (still have **not solved** this)
  - The number of unlabelled graphs on n nodes grows superexponentially: $|\mathcal{G}_n| \in \Theta\left(2^{\binom{n}{2}}/n!\right)$ (we have **solved** this!)

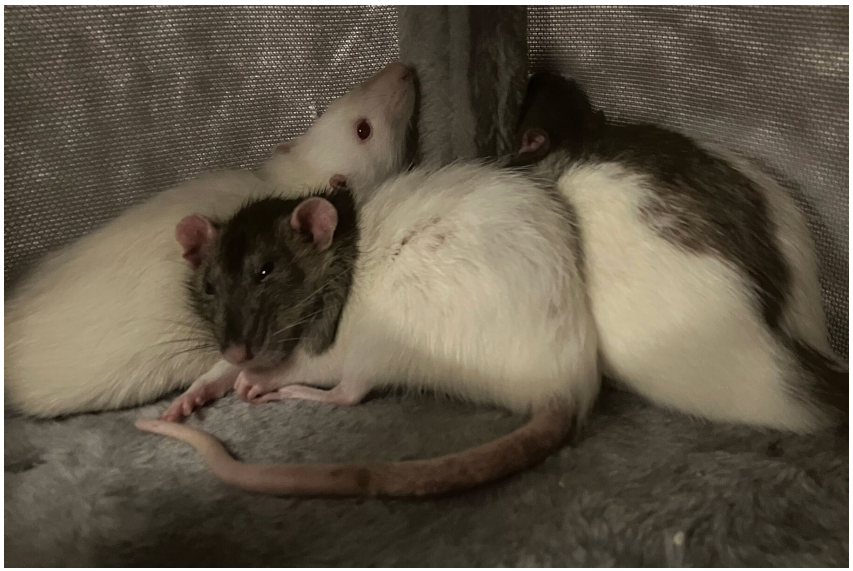


Figure: My friend Naomi's rats: **Parsnip** (left), **Arugula** (middle), and **Clove**.

Algorithm Setup (Part 1)

Design an $O\left(2.9558^n \cdot \sqrt{n}\right)$ algorithm to find all efficient connected networks for fixed (n, D, X). (Given all connected solutions up to n, can then check every **partition** $n = \sum n_i$ and compare sums $\sum \mathcal{V}^*_{n_i}$ of optimal values.)

- For $D = 2$: efficient $\iff$ n-cycle
  - So, restrict our attention to $n \geq 5$ and $D \geq 3$
- Only three things matter: **cycle size** m, shape of **pendant trees** $T_1, T_2, \ldots, T_m$ (including root cycle nodes $c_1, c_2, \ldots, c_m$; trivial trees count), and **positions** of trees on the cycle
  - So, try all possible cycle sizes and all possible ways to make pendant trees from the leftover nodes
- In an efficient connected network, **cycle size is bounded**:

$$m \leq \beta(n, D) := \begin{cases} \max\left(\lfloor (n+1)/2 \rfloor, 5\right) & \text{if } D = 3, \\ \max\left(\lfloor n/2 \rfloor, 5\right) & \text{if } D \geq 4. \end{cases}$$

Algorithm Setup (Part 2)

- Decompose a tree's contribution into its **internal value** $\mathcal{V}(T_i)$ and (some coefficient times) its **"value to the cycle"** $\varphi_{T_i}(c_i)$
- Generate (D, X, ℓ)**-rat trees**: rooted trees T on $\ell + 1$ nodes that
  - follow degree constraints: $(D-1)$-ary, with $\leq D-2$ nodes in level 1,
  - are not Pareto-dominated in terms of $\mathcal{V}(T)$ and $\varphi_T(\text{root})$.
  - **Loose** bound on the number of rat trees: $O\left(2.9558^{\ell+1} \cdot (\ell+1)^{-3/2}\right)$
- **Additional pruning:** as we go, branch-and-bound eliminates even more tree choices when building candidate networks
- Placing the trees is a **quadratic assignment problem** (QAP)
  - Finding **all** optimal assignments is **NP-hard**
  - But if **sorted** as $\varphi_{T_1}(c_1) \geq \varphi_{T_2}(c_2) \geq \cdots \geq \varphi_{T_m}(c_m)$, then $\cdots \sim c_5 \sim c_3 \sim c_1 \sim c_2 \sim c_4 \sim \cdots$ (i.e., a π^* assignment) is **a** solution
  - So, try to find all efficient connected networks up to cycle size and pendant trees, and just use π^*

Example: Pendant Tree QAP



Figure: The "more valuable" (to the cycle) a tree, the nearer to the middle it is.

Algorithm: Efficient Connected Networks

input: $n \geq 5, \quad D \in \{3, 4, \ldots, n-2\}, \quad X \in (0, 1).$

1. set $\Gamma \leftarrow \emptyset$; set $\mathcal{V}^* \leftarrow 0$.
2. **for each** $\ell = 0, 1, \ldots, n-3$ **do**
   a. compute and store $\tau_\ell \leftarrow \{(T, \mathcal{V}(T), \varphi_T(\text{root})) \mid T \text{ is } (D, X, \ell)\text{-rat}\}$.

   end for
3. **for each** $m = 3, 4, \ldots, \beta(n, D)$ **do**
   a. **for each** partition $(\ell_1, \ell_2, \ldots, \ell_m) \geq \vec{0}$ s.t. $n - m = \sum \ell_i$ **do**
      i. **for each** [unpruned, unordered m-tuple of data] $\in \tau_{\ell_1} \times \cdots \times \tau_{\ell_m}$ **do**
         A. set $G \leftarrow [m\text{-cycle with pendant trees } T_1, \ldots, T_m \text{ under } \pi^*]$.
         B. compute $\mathcal{V} \leftarrow \mathcal{V}(G)$ using stored $\mathcal{V}(T), \varphi_T(\text{root})$ data.
         C. **if** $\mathcal{V} > \mathcal{V}^*$ **then**; set $\Gamma \leftarrow \{(T_1, \ldots, T_m)\}$; set $\mathcal{V}^* \leftarrow \mathcal{V}$.
            else if $\mathcal{V} = \mathcal{V}^*$ **then**; set $\Gamma \leftarrow \Gamma \cup \{(T_1, \ldots, T_m)\}$.
            end if

      end for

   end for

end for

output: Γ (fully describes equivalence classes), $\mathcal{V}^*$ (optimal value).

Data and Implications

- In practice, **significantly faster**: scales as roughly $O(1.3^n)$
- Identified all efficient connected networks for $D \in \{3, 4, 5, 6\}$, $n \leq 36$, and $X \in \{0.01, 0.05, 0.1, 0.2, \ldots, 0.8, 0.9, 0.95, 0.99\}$ (**13 values**), as well as up to $n = 43$ for $D = 3$
- Identified interesting Nash networks, both efficient and inefficient (not in scope here, but helpful for our mathematical economics paper)
- Found empirical evidence (not yet proven) for:
  - **Efficient** $\implies$ **connected** (recursively checked integer partitions of n)
  - **Complete trees** (i.e., BFS-filled) maximize network value
  - **Cycle size** m appears to be **sublinear** in n
  - Again, a lot of interesting **Nash stuff** not covered here ...

Examples: Efficient Networks

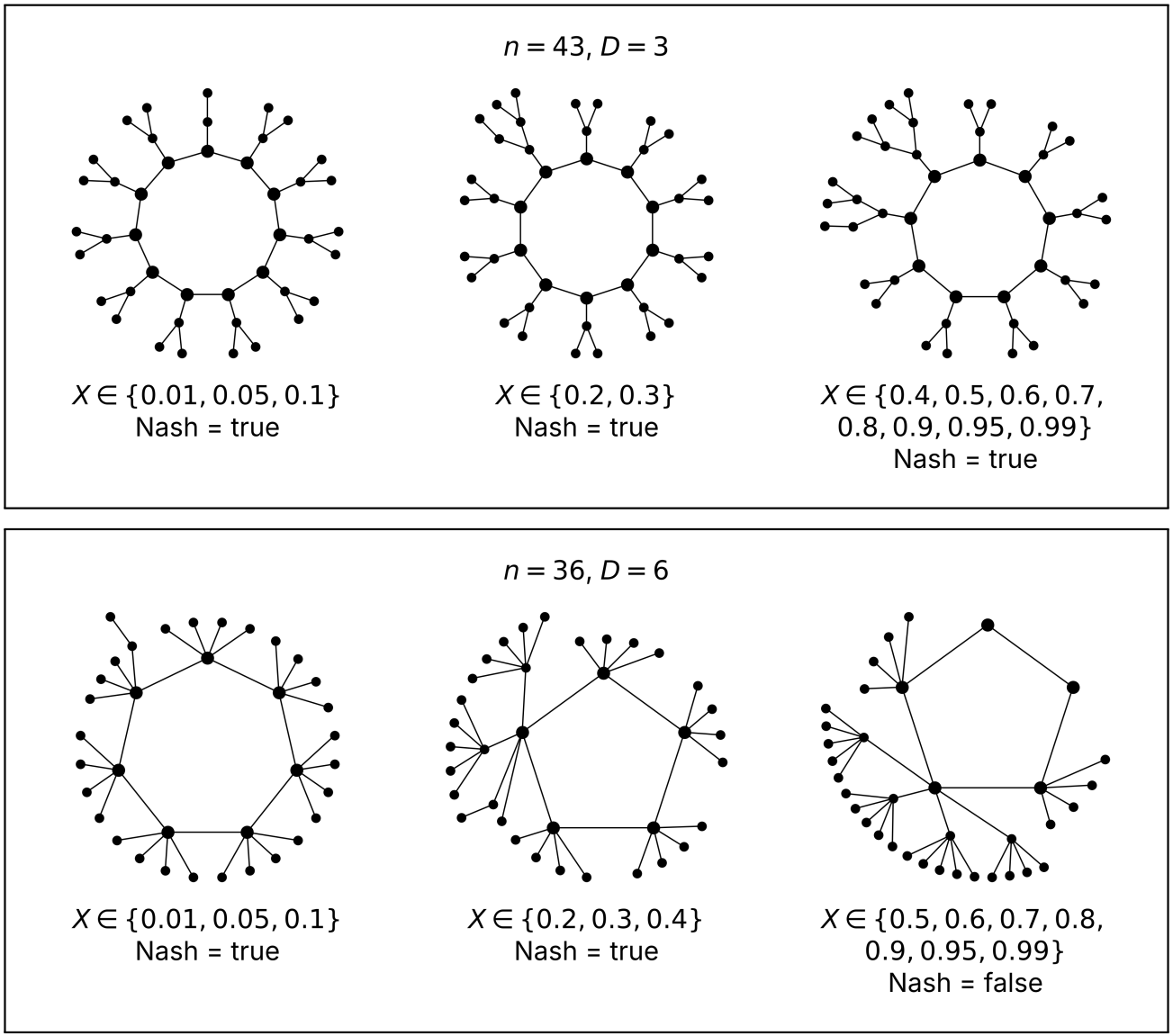


Figure: Efficient networks exhibiting connectedness and tree completeness.

Future Work

1. Use HPC to reach $n \approx 60$–80 for efficient network data
2. Finish **our paper**: *Unilateral link formation in degree-constrained communication networks*
3. See what can be proven from the empirical conjectures
4. Improve the $O\left(2.9558^n \cdot \sqrt{n}\right)$ complexity of the algorithm
5. Explore degree-constrained networks with **other strength topologies**



Figure: My cats, **Yuki** (left) and **Ash**, being sleepy and cute. <3

Thank you for listening!



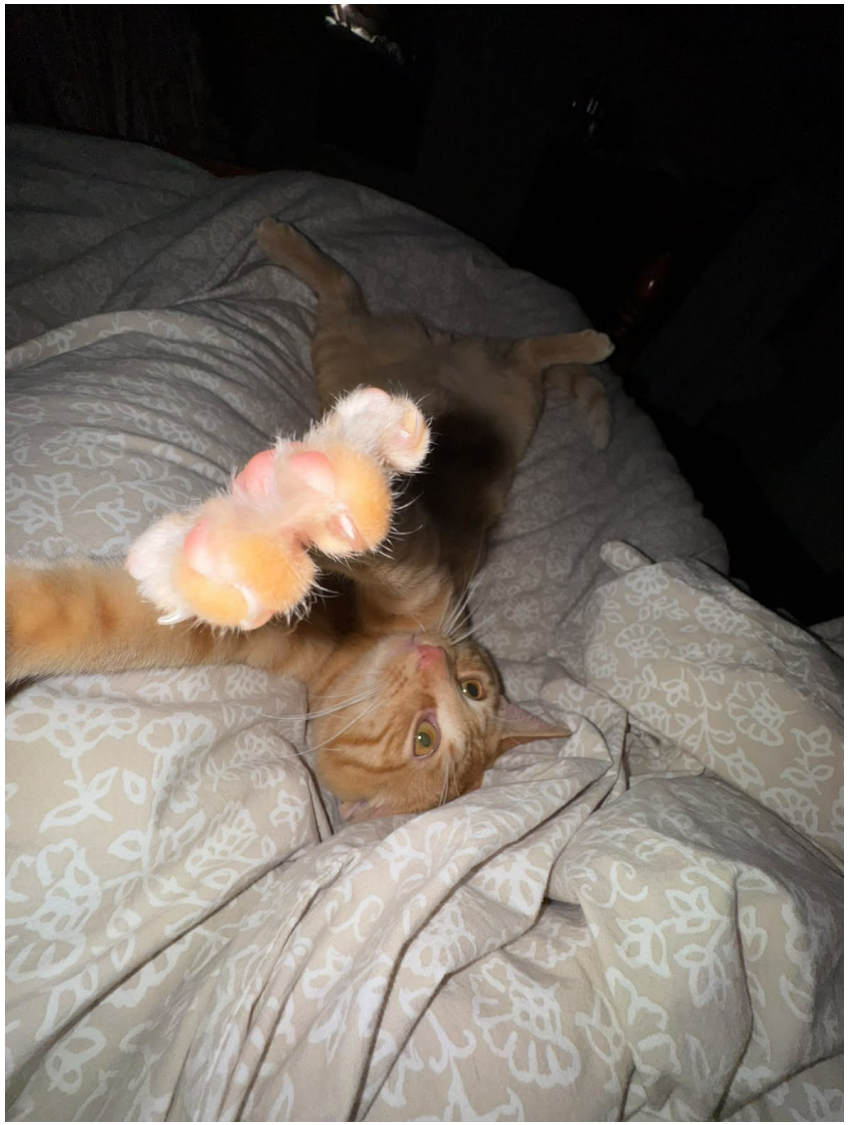


Figure: My friend Sasha's cats: **Milo** (left) and **Juno**.

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