Numerical Mode-Finding in Layered Media via Banded RRQR

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Land Acknowledgement

McMaster University stands on the traditional territory shared between the Haudenosaunee confederacy and the Anishinaabe nations. We must acknowledge a debt to those who were here before us, and recognize our responsibility, as guests, to respect and honour the intimate relationship Indigenous peoples have to this land.

This land acknowledgement, viewed through an intersectional lens, recognizes that the same forces of colonial violence that occupied and settled Turtle Island governed in a way that enacted, and continues to enact, violence upon queer, trans, and two-spirit peoples. We recognize that we cannot adequately support queer, trans, and two-spirit peoples without also supporting action towards reconciliation with Indigenous peoples.

What Is Mode-Finding?

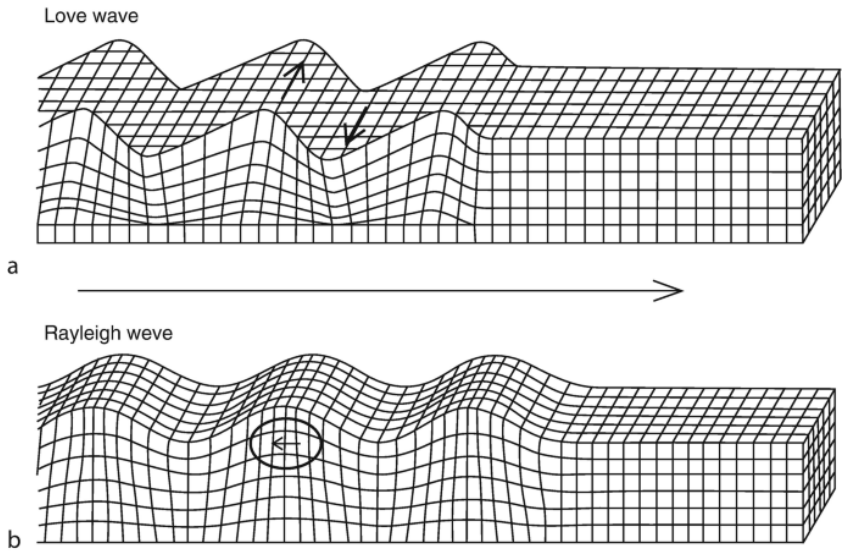
- **Waves** propagating through a layered medium (e.g., **soil** or **star shells**) "echo" back and forth at layer boundaries
- For a wave to persist, these echoes all need to line up with each other
- The specific **frequencies** ω (and sometimes **speeds** c) at which this happens are called **modes**
- For each input ω or (ω, c), we construct a **stiffness matrix** K
- Inputs that make this matrix singular (i.e., send certain nonzero vectors in its domain to $\mathbf{0}$) correspond to modes
- Each boundary gets some fixed number of rows in the matrix representing equations forcing consistency in forces/motion



Figure: A **black bear** I spotted in Jasper, AB back in July 2018.

Applications in Geophysics

- Geophysicists often use waves sent from the surface to probe deep into the Earth
- e.g., for **subsurface imaging** and **earthquake predictions**
- These waves have both frequency and speed (how fast they travel horizontally), so the input is (ω, c)
- There are **Love waves** ("side to side," bandwidth $b = 1$) and **Rayleigh waves** ("rolling," $b = 3$)
- The test case I use is a synthetic 100-layer profile of each type



Applications in Geophysics

Love wave stiffness matrix:

$$K(\omega, c) = \begin{bmatrix} \mu_1\nu_1 \coth(\nu_1 h_1) & -\mu_1\nu_1 \operatorname{csch}(\nu_1 h_1) & & \\ -\mu_1\nu_1 \operatorname{csch}(\nu_1 h_1) & \sum_{j=1}^{2} \mu_j\nu_j \coth(\nu_j h_j) & \ddots & \\ & \ddots & \ddots & -\mu_N\nu_N \operatorname{csch}(\nu_N h_N) \\ & & -\mu_N\nu_N \operatorname{csch}(\nu_N h_N) & \mu_N\nu_N \coth(\nu_N h_N) + \mu_{N+1}\nu_{N+1} \end{bmatrix}$$

$$\nu_j := \omega \sqrt{\frac{1}{c^2} - \frac{1}{V_{s,j}^2}}$$

Applications in Geophysics

Rayleigh wave stiffness matrix:

$$K(\omega, c) = \begin{bmatrix} A_1 & B_1 & & \\ C_1 & D_1 + A_2 & \ddots & \\ & \ddots & \ddots & B_N \\ & & C_N & D_N + K_{N+1} \end{bmatrix}$$

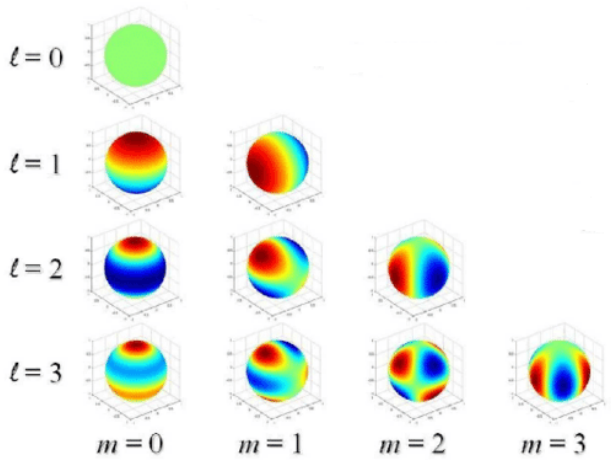
$$k := \omega/c, \quad \gamma_j := k^2 + s_j^2, \quad r_j := \sqrt{k^2 - \omega^2/V_{p,j}^2}, \quad s_j := \sqrt{k^2 - \omega^2/V_{s,j}^2},$$

$$E_j := \begin{bmatrix} ik & 0 & 0 & s_j \\ 0 & ir_j & k & 0 \\ ik\cosh(r_j h_j) & ik\sinh(r_j h_j) & s_j \sinh(s_j h_j) & s_j \cosh(s_j h_j) \\ ir_j \sinh(r_j h_j) & ir_j \cosh(r_j h_j) & k\cosh(s_j h_j) & k\sinh(s_j h_j) \end{bmatrix}, \quad F_j := \mu_j \begin{bmatrix} 0 & -2ikr_j & -\gamma_j & 0 \\ -i\gamma_j & 0 & 0 & -2ks_j \\ 2ikr_j \sinh(r_j h_j) & 2ikr_j \cosh(r_j h_j) & \gamma_j \cosh(s_j h_j) & \gamma_j \sinh(s_j h_j) \\ i\gamma_j \cosh(r_j h_j) & i\gamma_j \sinh(r_j h_j) & 2ks_j \sinh(s_j h_j) & 2ks_j \cosh(s_j h_j) \end{bmatrix},$$

$$\begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} := F_j E_j^{-1}, \quad K_{N+1} := \frac{\mu_{N+1}}{k^2 - r_{N+1}s_{N+1}} \begin{bmatrix} r_{N+1}\left(k^2 - s_{N+1}^2\right) & k\left(2r_{N+1}s_{N+1} - \gamma_{N+1}\right) \\ k\left(2r_{N+1}s_{N+1} - \gamma_{N+1}\right) & s_{N+1}\left(k^2 - s_{N+1}^2\right) \end{bmatrix}$$

Applications in Astrophysics

- Stars **oscillate** (a.k.a. "vibrate") at certain valid frequencies[1]
- This depends only on frequency, so the input is just ω
- Measuring these frequencies lets astrophysicists[2] "see" inside a star
- The test case I use is $\ell = 1$ (i.e., two halves oscillate opposite each other)[3] for the **red giant HD 128279**, with bandwidth $b = 2$



[1] I am a political scientist and computer scientist, **not** a physicist. Despite my best friend's efforts—you should check out her talk—I do not know how to explain this. I just know the math!

[2] Like my awesome bestie. Seriously, she's so fed up with my cluelessness right now.

[3] My bestie vehemently insisted I "do the hand motions" here, so I will do so.

Applications in Astrophysics

Stellar oscillation matrix (Cowling approximation):

$$K(\omega) = \begin{bmatrix} \omega^2 r_0 & -\ell/\rho_0 & & & \\ -\left(I + \frac{h_0}{2} A_0\right) & I - \frac{h_0}{2} A_1 & & & \\ & \ddots & \ddots & & \\ & & & -\left(I + \frac{h_{M-1}}{2} A_{M-1}\right) & I - \frac{h_{M-1}}{2} A_M \\ & & & -\rho_M g_M & 1 \end{bmatrix}$$

$$A_i := \begin{bmatrix} -\dfrac{2}{r_i} + \dfrac{g_i}{c_{s,i}^2} & \dfrac{1}{\rho_i c_{s,i}^2}\left(\dfrac{S_{\ell,i}^2}{\omega^2} - 1\right) \\ \rho_i\left(\omega^2 - N_i^2\right) & -\dfrac{g_i}{c_{s,i}^2} \end{bmatrix}$$

Problems with Usual Methods

- $O(nb^2)$ **determinant scanning**: looks for sign changes in $\det(K)$ as input varies (most common standard)
  - Two nearby modes can cancel sign changes, evading detection
  - Does not return **mode shape** (i.e., the kernel vector associated with each mode, representing how much each layer boundary moves)
- $O(n^3)$ **dense SVD or RRQR**: normal SVD/RRQR routines
  - Numerically robust but slow
- $O(nb^2)$ **sparse SVD**: SVD optimized for sparse matrices
  - Faster but (in theory for linear algebra reasons) may converge slower when modes are close by[4]

[4] I have not yet tested out sparse SVD; this project originated as a numerical linear algebra one, with the intent of developing a banded RRQR algorithm. This talk just applies the algorithm to physics cases, but it is very possible that sparse SVD is a competitive alternative.

Banded RRQR as an Alternative

- **Rank-revealing QR** factors $KP = QR$ where Q is orthonormal, R is upper triangular, and P reorders columns, choosing P so the most linearly independent columns come first
- A near-zero last diagonal entry of R **reveals** that K is singular (i.e., its producing inputs form a mode)
- Back-substitution gives the kernel vector or **mode shape**
- Ordinary RRQR is $O(n^3)$, and sparse QR is only heuristically rank-revealing
- **Banded RRQR** only pivots within the current block of columns representing one boundary's requirements, taking $O(nb^2)$ time
- This is done "from both ends" of K

Numerical Results

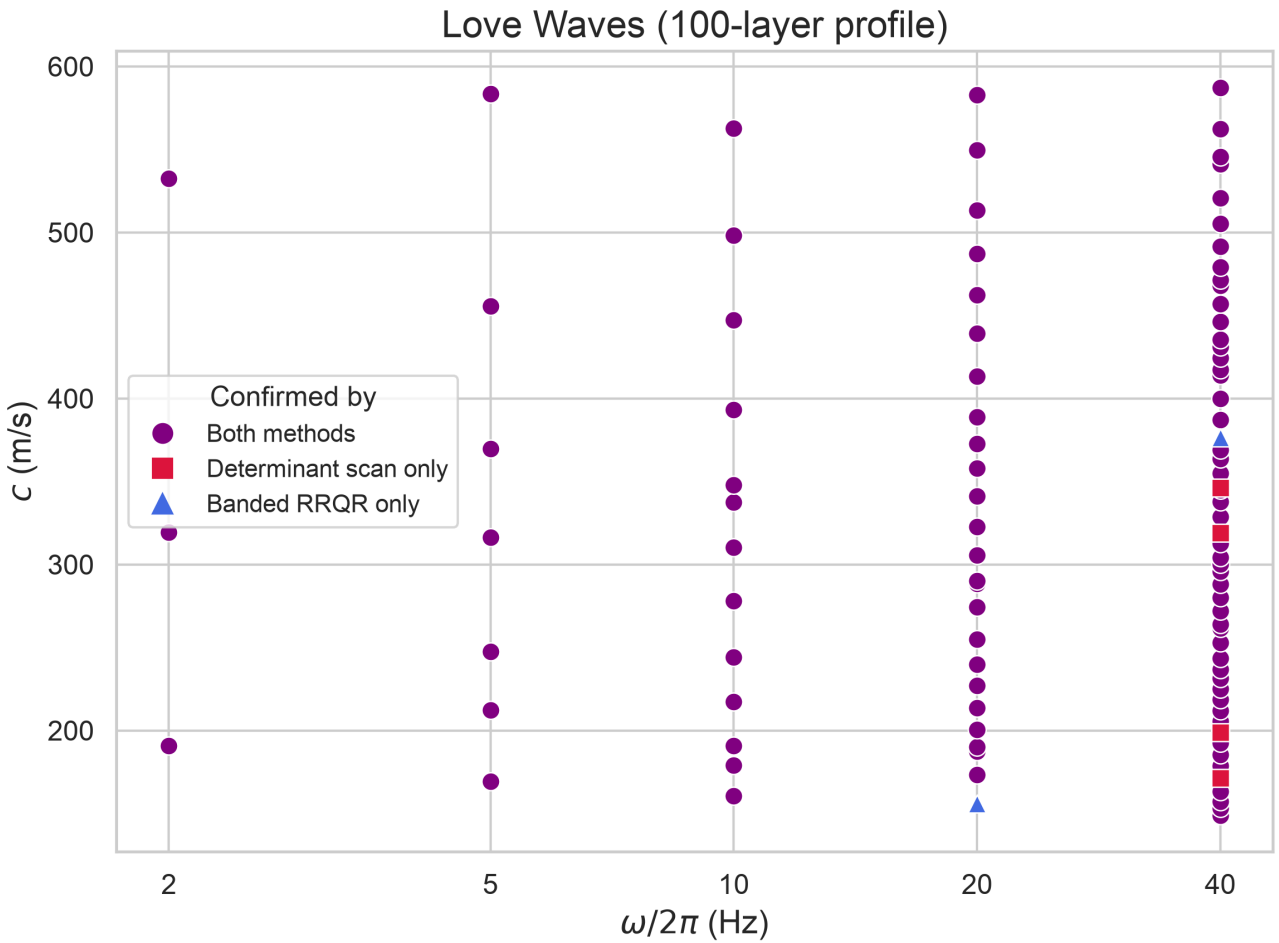


Figure: Banded RRQR finds 101/105 and determinant scanning finds 103/105. However, banded RRQR's misses come from "merging" nearby modes whereas determinant scanning misses pairs of close modes altogether.

Numerical Results

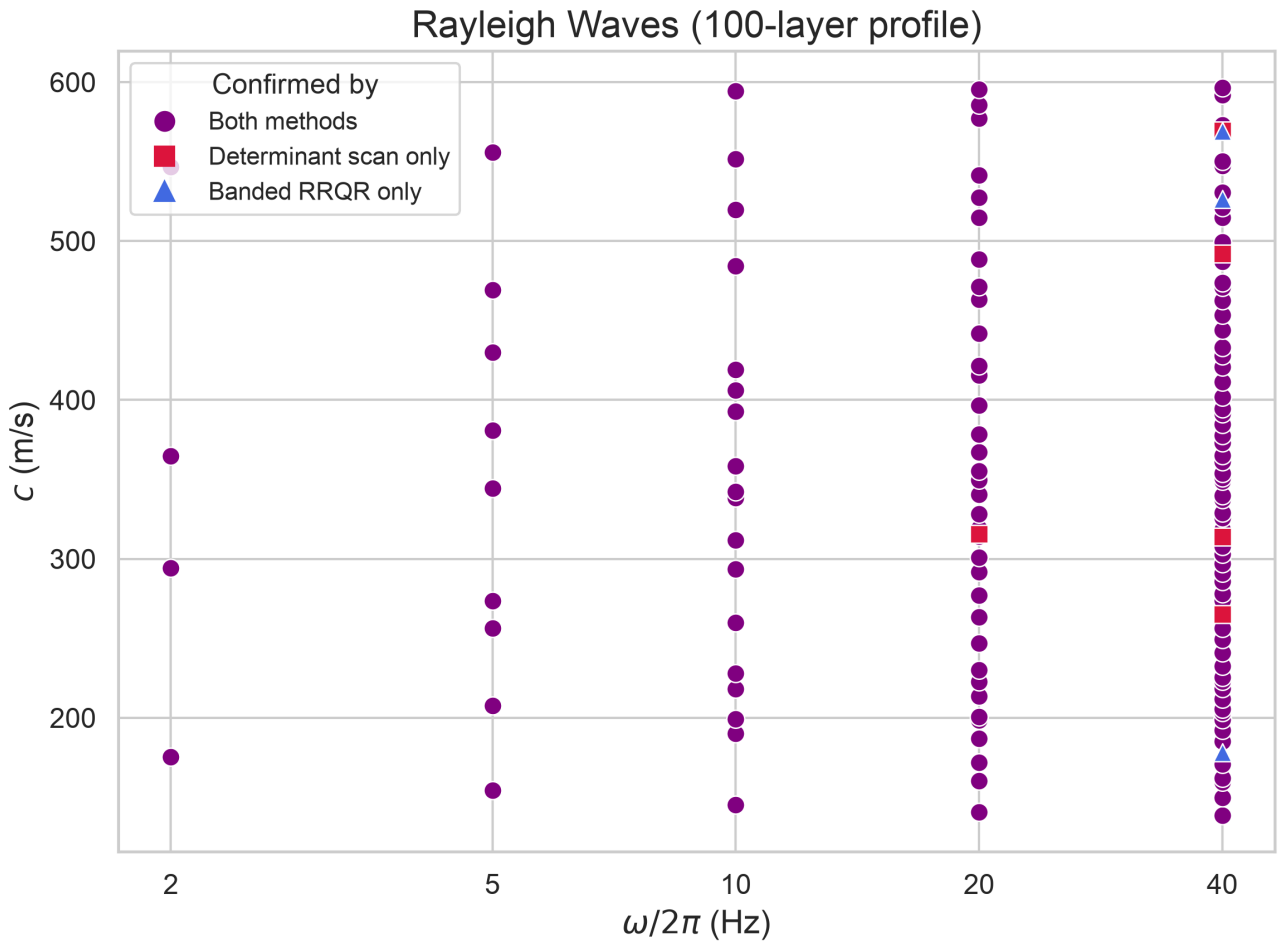


Figure: Again, banded RRQR finds 133/138 while determinant scanning finds 135/138, and banded RRQR "merges" while determinant scanning misses nearby modes entirely.

Numerical Results

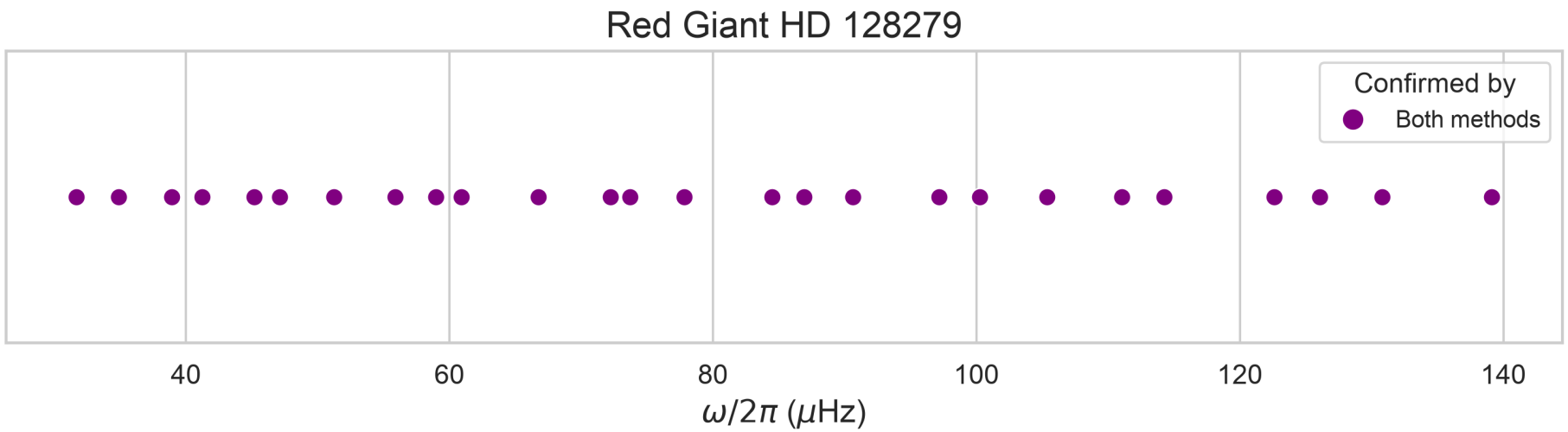
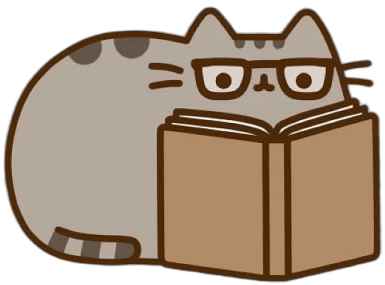


Figure: Both banded RRQR and determinant scanning find exactly 26 modes for red giant HD 128279 with $\ell = 1$.



Future Work

- **Damping:** Account for energy absorption in soil (right now, pretending no energy is ever lost)
- **Other stellar oscillations:** Check, e.g., other stars or other ℓ parameters
- **Stop merging:** Correctly separate nearby modes
- **Check alternatives:** Compare directly against sparse SVD (both w.r.t. runtime and robustness of results)
- **Further improve the algorithm:** Began primarily as a numerical analysis/algorithms project,[5] so focus on the actual algorithm itself

[5] And on the off chance this is ever published (most of my research nowadays is in political philosophy or graph algorithms), it will be in an NA journal, not a physics one.

Thank you for listening!



Figure: My cats, **Yuki** (left) and **Ash**, being sleepy and cute.