Spectral Heuristics for k-Incoherent Decompositions

Luis M. B. Varona[1,2,3]

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[1] *Department of Politics & International Relations, Mount Allison University*
[2] *Department of Economics, Mount Allison University*
[3] *Department of Mathematics & Computer Science, Mount Allison University*

Quantum Computing and Information

- Quantum computers use **qubits**, not classical **bits** (0 or 1)
- A qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is a **superposition** of "0" and "1," where $\alpha, \beta \in \mathbb{C}$ are probability amplitudes with $|\alpha|^2 + |\beta|^2 = 1$
- Quantum computers can solve (certain) problems far faster, but are highly susceptible to **decoherence**
- Recall the thought experiment of **Schrödinger's cat(s)**:

Figure 1: My cats, **Yuki** (left) and **Ash**, may be both **awake** and **asleep** ...

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The Density Matrix of a System

Definition (Density matrix)

The **density matrix** $\rho \in \mathbb{C}^{n \times n}$ of some quantum system with n states is an $n \times n$ Hermitian (i.e., $\rho = \rho^\dagger$) matrix such that:

- Each diagonal entry $\rho_{i,i} \in \mathbb{R}$ is the **probability** of being in state i
- Each off-diagonal entry $\rho_{i,j} \in \mathbb{C}$ (rather, its absolute value) is the **coherence** between states i and j

- **e.g.:** a **pure** qubit with a 75% probability of being in $|0\rangle$ and a 25% probability of being in $|1\rangle$ gives

$$\rho = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2}e^{i\phi} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2}e^{-i\phi} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4}e^{-i\phi} \\ \frac{\sqrt{3}}{4}e^{i\phi} & \frac{1}{4} \end{bmatrix},$$

  where $e^{i\phi} \in U(1)$ is some arbitrary complex phase factor
- Purity yields **coherence** (if $|0\rangle, |1\rangle$ are chosen wisely), enabling parallel information processing impossible for classical systems

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k-Incoherence in Mixed States

- Most states are less coherent **mixed** combinations of pure states—studying **k-incoherence** lets us utilize them anyway

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A density matrix $\rho \in \mathbb{C}^{n \times n}$ is said to be **k-incoherent** for some $k \in \{1, 2, \ldots, n\}$ if there exists a collection of pure state vectors $|v_1\rangle, |v_2\rangle, \ldots, |v_r\rangle \in \mathbb{C}^n$ with weights $c_1, c_2, \ldots, c_r > 0$ such that:

- Each $|v_i\rangle$ has at most k nonzero entries
- The weights sum to 1: $\sum_{i=1}^{r} c_i = 1$
- The density matrix can be written as $\rho = \sum_{i=1}^{r} c_i |v_i\rangle\langle v_i|$

- **k-incoherent decompositions** tell us which k-subsets of states cohere, allowing us to exploit their quantum resources
- Decompositions with **fewer terms** are desirable, since they demand fewer pure states to be prepared in a lab

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An Example Decomposition

- Consider the following mixed state:

$$\rho = \frac{1}{6}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

- The **minimum** k for k-incoherence is 2, and the **minimum number of terms** for a 2-decomposition is 3:

$$\rho = \frac{1}{3}|v_1\rangle\langle v_1| + \frac{1}{3}|v_2\rangle\langle v_2| + \frac{1}{3}|v_3\rangle\langle v_3|,$$

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Using Semidefinite Programming

- Traditionally, we apply **semidefinite programming** (where linear programming $\subset$ **SDP** $\subset$ convex optimization) to check for k-incoherence and, if applicable, find a decomposition
- SDP is guaranteed to find a decomposition if one exists, but uses far more pure states (expensive to prepare!) than needed
- **e.g.:** SDP (with the Clarabel solver) produces a **12-term** 2-incoherent decomposition of the density matrix

$$\frac{1}{9.8581}\begin{bmatrix} 1.722 & 0.0957 & -0.3164 & 0.0898 \\ 0.0957 & 0.5183 & -0.6569 & -0.1482 \\ -0.3164 & -0.6569 & 5.2127 & 0.4955 \\ 0.0898 & -0.1482 & 0.4955 & 2.4051 \end{bmatrix},$$

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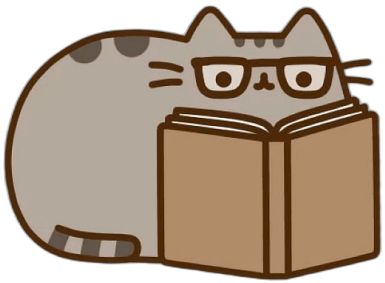
4. If failure: use **cyclic coordinate descent** to refine our choice of each $\mathbf{v}_i$, hoping to eventually get $\rho = \sum_{i=1}^{r} \mathbf{v}_i \mathbf{v}_i^\dagger$ exactly



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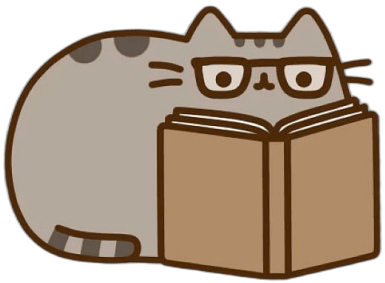

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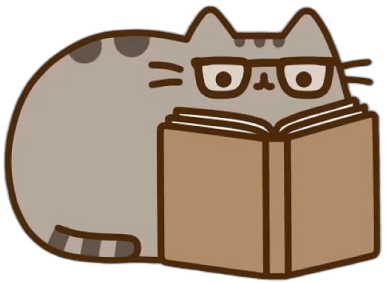

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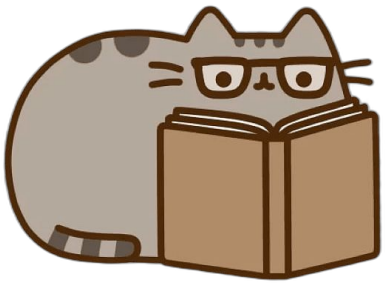

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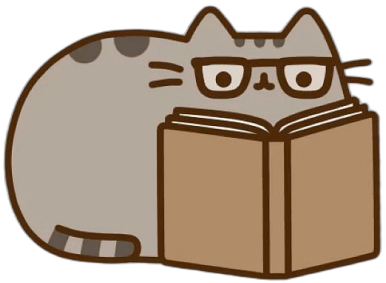

SDP vs. Heuristic Methods

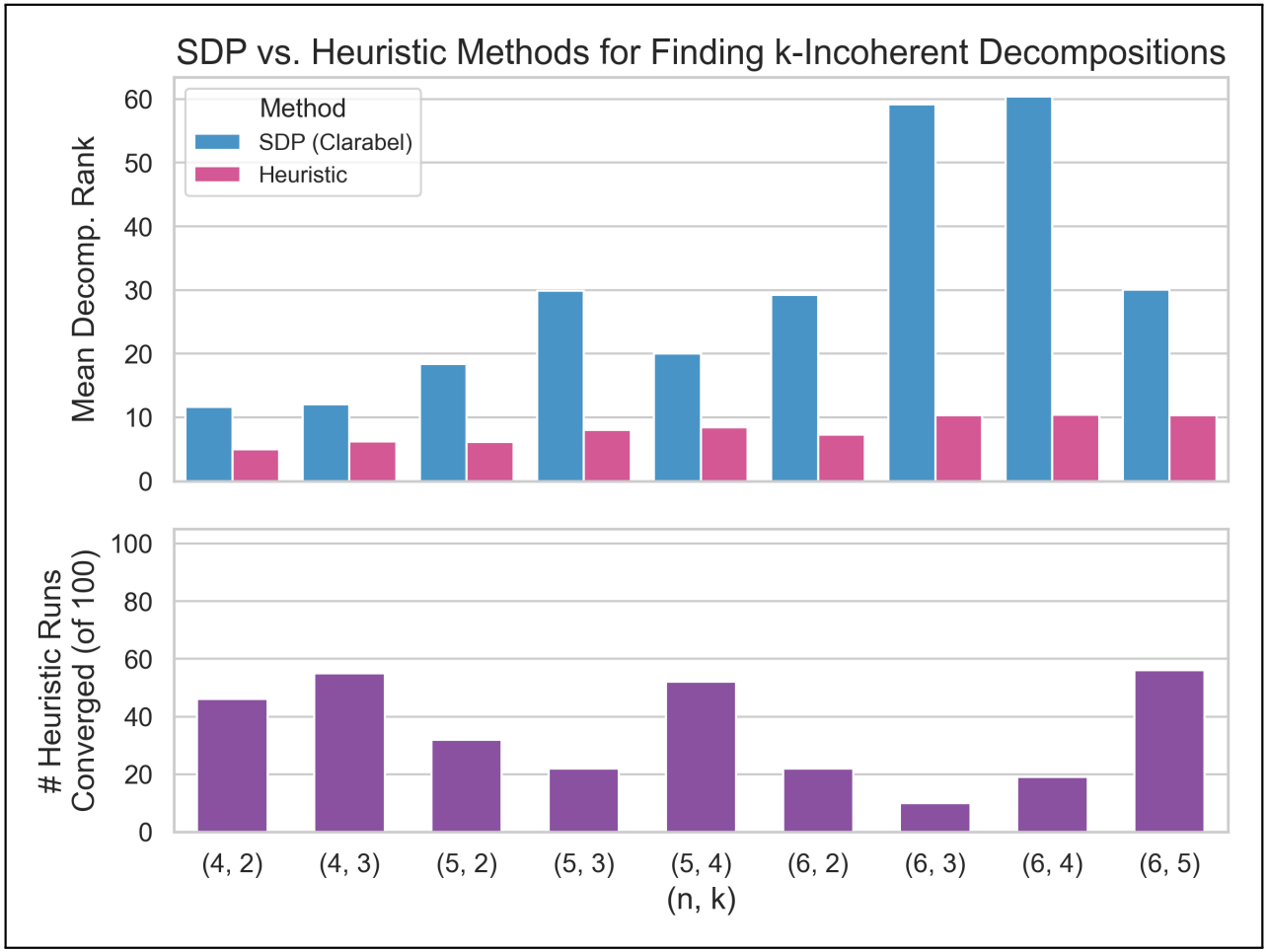

Figure 2: The maximum-eigenvalue heuristic with cyclic coordinate descent converges $\approx \mathbf{35}\%$ of the time to within Frobenius distance $< 10^{-14}$, requiring only $\approx \mathbf{32.3}\%$ as many terms as SDP when successful.

Thank you for listening!

Figure 3: A **barred owl** in Midgic, NB—approximately two hours away from Charlottetown, PE when travelling at the maximum air-speed velocity of an unladen owl. [Credit: **Jaden Barney**.]